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ON THE CONVERGENCE OF SOME "IRREGULARLY" OSCILLATING SERIES

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On the convergence of some "irregularly" oscillating series

by

J. van de Lune

ABSTRACT

A solution of advanced problem 6105, Amer. Math. Monthly, Vol. 83, No 7 (1976) 573 (proposed by H.D. RUDERMAN).

KEY WORDS & PHRASES: *Series, Uniform Distribution.*

0. INTRODUCTION

In [5] RUDERMAN proposed the following (unsolved) problem: Prove the convergence of the series

$$(0.1)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{[n\sqrt{2}]}}{n}$$

and estimate its sum.

In this note it will be shown, more generally, that if s is any positive real number and if α is any real quadratic irrationality then the series

$$(0.2)\qquad \sum_{n=1}^{\infty} \frac{(-1)^{[n\alpha]}}{n^s}$$

is convergent. Also it will be shown that the sum S of (0.1) satisfies

$$(0.3)\qquad -.51545 < S < -.51538 .$$

1. CONVERGENCE

Throughout this note s will be some fixed positive real number whereas α denotes some fixed real quadratic irrationality (unless stated explicitly otherwise).

<u>THEOREM</u> 1. *The series* (0.2) *converges.*

<u>PROOF</u>. Defining

$$(1.1)\qquad \begin{cases} \sigma_0 = 0 \\ \sigma_n = \sum_{k=1}^{n} (-1)^{[k\alpha]}, & (n \in \mathbb{N}) \end{cases}$$

and

$$(1.2)\qquad S_n = \sum_{k=1}^{n} \frac{(-1)^{[k\alpha]}}{k^s}, \qquad (n \in \mathbb{N})$$

we have (by partial summation)

$$(1.3) \qquad S_n = \frac{\sigma_n}{n^s} + \sum_{k=1}^{n-1} \sigma_k \left\{ \frac{1}{k^s} - \frac{1}{(k+1)^s} \right\} .$$

Now suppose that we can prove that

$$(1.4) \qquad \sigma_n = O(\log n), \qquad (n \to \infty).$$

Then we have

$$(1.5) \qquad \lim_{n\to\infty} \frac{\sigma_n}{n^s} = 0$$

and (by the mean value theorem)

$$(1.6) \qquad \sigma_k \left\{ \frac{1}{k^s} - \frac{1}{(k+1)^s} \right\} = \sigma_k (k - (k+1)) \frac{-s}{(k+v)^{s+1}}$$

for some $v \in (0,1)$, so that

$$(1.7) \qquad \sigma_k \left\{ \frac{1}{k^s} - \frac{1}{(k+1)^s} \right\} = O\left(\frac{\log k}{k^{s+1}} \right), \qquad (k \to \infty).$$

From (1.3), (1.5) and (1.7) it follows that (0.2) converges:

$$(1.8) \qquad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} \sigma_k \left\{ \frac{1}{k^s} - \frac{1}{(k+1)^s} \right\},$$

the last series being absolutely convergent. Hence, in order to prove the convergence of (0.2) it suffices to show that (1.4) holds true.

In order to prove (1.4) we observe that

$$(1.9) \qquad [-x] + [x] = -1, \qquad (x \in \mathbb{R} \setminus \mathbb{Z})$$

from which it is easily seen that (without loss of generality) we may assume $\alpha > 0$. A simple counting proces reveals that if $\alpha > 0$ then

$$(1.10) \qquad \sigma_n = \sum_{k=1}^{M} (-1)^{k-1} \left\{ \left[\frac{k}{\alpha} \right] - \left[\frac{k-1}{\alpha} \right] \right\} + (-1)^M \left\{ n - \left[\frac{M}{\alpha} \right] \right\} ,$$

where $M = [n\alpha]$. Since

$$(11.a) \qquad n - \left[\frac{M}{\alpha}\right] = \frac{n\alpha}{\alpha} - \left[\frac{[n\alpha]}{\alpha}\right] \ge \frac{n\alpha - [n\alpha]}{\alpha} > 0$$

and

$$(1.11b) \qquad n - \left[\frac{M}{\alpha}\right] \le n - \frac{M}{\alpha} + 1 = \frac{n\alpha - [n\alpha]}{\alpha} + 1 < \frac{1+\alpha}{\alpha}$$

we have

$$(1.12) \qquad \sigma_n = \sigma_n^* + O(1), \qquad (n \to \infty)$$

where

$$(1.13) \qquad \sigma_n^* \stackrel{\text{def}}{=\!=} \sum_{k=1}^{M} (-1)^{k-1}\left\{\left[\frac{k}{\alpha}\right] - \left[\frac{k-1}{\alpha}\right]\right\} .$$

Hence, it suffices to show that

$$(1.14) \qquad \sigma_n^* = O(\log n), \qquad (n \to \infty).$$

Writing β instead of $\frac{1}{\alpha}$ we have

$$(1.15) \qquad \sigma_n^* = \sum_{k=1}^{M} (-1)^{k-1}\{[k\beta] - [(k-1)\beta]\} =$$

$$= 2\sum_{k=1}^{M} (-1)^{k-1}[k\beta] + (-1)^M[M\beta] =$$

$$= (-1)^M[M\beta] + 2\{[\beta] - [2\beta] + [3\beta] - + \ldots + (-1)^{M-1}[M\beta]\} =$$

$$= (-1)^M[M\beta] + 2\{[\beta] + [2\beta] + \ldots + [M\beta]\} +$$

$$- 4\left\{[2\beta] + [4\beta] + \ldots + \left[2\left[\frac{M}{2}\right]\beta\right]\right\} =$$

$$= (-1)^M[M\beta] - 2\sum_{k=1}^{M} \{k\beta - [k\beta] - \tfrac{1}{2}\} + 4\sum_{k=1}^{[M/2]} \{2k\beta - [2k\beta] - \tfrac{1}{2}\} +$$

$$+ 2\sum_{k=1}^{M} (k\beta - \tfrac{1}{2}) - 4\sum_{k=1}^{[M/2]} (2k\beta - \tfrac{1}{2}) =$$

$$= -2R_M^*(\beta) + 4R_{[M/2]}^*(2\beta) + \Delta_M$$

where (compare [2; p.98])

(1.16) $$R_n^*(\beta) \overset{\text{def}}{=\!=} \sum_{k=1}^{n} (k\beta - [k\beta] - \tfrac{1}{2})$$

and

(1.17) $$\Delta_M = (-1)^M[M\beta] + 2\sum_{k=1}^{M} (k\beta-\tfrac{1}{2}) - 4\sum_{k=1}^{[M/2]} (2k\beta-\tfrac{1}{2}).$$

One may verify that if $a \in \mathbb{N}$ then

(1.18) $$\Delta_{2a} = [2a\beta] - 2a\beta$$

and

(1.19) $$\Delta_{2a-1} = (2a-1)\beta - [(2a-1)\beta] + \beta - 1$$

so that

(1.20) $$\Delta_M = O(1), \qquad (n \to \infty).$$

Now we recall a theorem of LERCH (c.f. [3] or [2; p.102]) saying that if β has the regular continued fraction expansion

(1.21) $$\beta = \{b_0;b_1,b_2,b_3,\ldots\}$$

with $b_i \le K = K(\beta)$ for all $i \in \mathbb{N}$, then

(1.22) $$R_n^*(\beta) = O(\log n), \qquad (n \to \infty).$$

Since α is a real quadratic irrationality, LERCH's theorem applies to $\beta = \frac{1}{\alpha}$ as well as to 2β. Hence, from (1.15) and (1.20) we obtain that

(1.23) $$\sigma_n^* = O(\log M) + O\left(\log\left[\frac{M}{2}\right]\right) + O(1) = O(\log n),$$

completing the proof of theorem 1.

REMARKS.

1. In OSTROWSKI [4; p.84-85] it is shown that if $\beta = \{b_0;b_1,b_2,b_3,...\}$ with $b_i \le K = K(\beta)$ for all $i \in \mathbb{N}$, then

$$2\beta = \{c_0;c_1,c_2,c_3,...\}$$

with $c_i \le 4K(\beta)$ for all $i \in \mathbb{N}$.
From this result and the proof of theorem 1 it follows that theorem 1 also holds true for all irrational α such that

$$\alpha = \{a_0;a_1,a_2,a_3,...\}$$

with $a_i \le K(\alpha)$ for all $i \in \mathbb{N}$.

2. In KHINTCHINE [1] it is shown that for any $\varepsilon > 0$ we have

$$R_n^*(\theta) = O\left((\log n)^{1+\varepsilon}\right), \qquad (n \to \infty)$$

for almost all $\theta \in \mathbb{R}$.
From this result and the proof of theorem 1 it follows that theorem 1 holds true for almost all $\alpha \in \mathbb{R}$.

2. THE SUM OF (0.1)

First we recall a theorem of OSTROWSKI (c.f. [4; p.81] or [2; p.103]) saying that if β has the regular continued fraction expansion $\beta = \{b_0;b_1,b_2,b_3,...\}$ with $b_i \le K$ for all $i \in \mathbb{N}$, then

$$(2.1) \qquad |R_n^*(\beta)| \le \frac{3}{2} K \log n, \qquad (n > 10).$$

In our case we have

$$(2.2) \qquad \beta = \tfrac{1}{2}\sqrt{2} = \{0;1,2,2,2,...\} = \{0;1,\overline{2}\}$$

so that $K(\beta) = 2$, and

$$(2.3) \qquad 2\beta = \sqrt{2} = \{1;2,2,2,...\} = \{1;\overline{2}\}$$

so that also $K(2\beta) = 2$.

Reexamining the proof of theorem 1 it is easily seen that for $M \geq 22$ and $\alpha = \sqrt{2}$ we have

$$(2.4) \qquad |\sigma_n| \leq \left[\frac{1}{\alpha}\right] + 1 + |\sigma_n^*| \leq$$

$$\leq 1 + |\Delta_M| + 2|R_M^*(\beta)| + 4|R_{[M/2]}^*(2\beta)| \leq$$

$$\leq 1 + 1 + 2.\frac{3}{2}\ 2 \log M + 4.\frac{3}{2}\ 2 \log \left[\frac{M}{2}\right] \leq$$

$$\leq 2 + 6 \log M + 12 \log \frac{M}{2} =$$

$$= 2 - 12 \log 2 + 18 \log M \leq$$

$$\leq 2 - 12 \log 2 + 18 \log n + 18 \log \sqrt{2} =$$

$$= 2 - 3 \log 2 + 18 \log n.$$

As a consequence we have

$$(2.5) \qquad S = \sum_{n=1}^{\infty} \frac{(-1)^{[n\alpha]}}{n} =$$

$$= \sum_{n=1}^{\infty} \frac{\sigma_n}{n(n+1)} = \sum_{n=1}^{N} \frac{\sigma_n}{n(n+1)} + \omega(N)$$

where, for $N \geq 15$, $(M \geq 22)$,

$$(2.6) \qquad |\omega(N)| \leq \sum_{n=N+1}^{\infty} \frac{|\sigma_n|}{n(n+1)} \leq$$

$$\leq \sum_{n=N+1}^{\infty} \frac{2 - 3 \log 2 + 18 \log n}{n(n+1)} =$$

$$= \frac{2 - 3 \log 2}{N + 1} + 18 \sum_{n=N+1}^{\infty} \frac{\log n}{n(n+1)} .$$

Observing that

$$(2.7)\qquad \sum_{n=N+1}^{\infty} \frac{\log n}{n(n+1)} < \sum_{n=N+1}^{\infty} \frac{\log n}{n^2} < \int_N^{\infty} \frac{\log x}{x^2}\, dx =$$

$$= \int_{\log N}^{\infty} e^{-u}\, u\, du = \frac{1 + \log N}{N}$$

we thus have

$$(2.8)\qquad |\omega(N)| < \frac{2 - 3\log 2}{N+1} + 18\,\frac{1+\log N}{N}\,.$$

One may verify that

$$(2.9)\qquad |\omega(10^7)| < .000\;031$$

and

$$(2.10)\qquad \sum_{n=1}^{10^7} \frac{\sigma_n}{n(n+1)} = -.515\;418\;4\;\ldots\,,$$

from which it follows that

$$(2.11)\qquad -.51545 < S < -.515\;38.$$

REMARK. Numerical work performed by H.J.J. TE RIELE suggests strongly that

$$S = -.515\;418\;4\;\ldots$$

with an accuracy of 6D.

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